MA125-6C, CALCULUS I

Test 3, March 18, 2015

Name (Print last name first): ...Key...

Show all your work and justify your answer!

No partial credit will be given for the answer only!

PART I

You must simplify your answer when possible.
All problems in Part I are 7 points each.

1. If $F(x) = \int_2^x (t^2+2)^{\frac{2}{3}}\,dt$, find the derivative $F'(x)$.

$F'(x) = (x^2+2)^{2/3}$ by the Fund. Thm of Calculus.

2. Use a Riemann sum with $n = 4$ terms and the right endpoint rule to approximate $\int_1^3 x^2\,dx$. (You don't need to multiply or add the terms.)

$\Delta x = \frac{3-1}{4} = \frac{1}{2}$

$x_0 = 1$

$x_i = 1 + i\,\Delta x$
$= 1 + \frac{i}{2}$

$f(x) = x^2$

$R_4 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$

$= \left(\frac{3}{2}\right)^2\left(\frac{1}{2}\right) + (2)^2\left(\frac{1}{2}\right) + \left(\frac{5}{2}\right)^2\left(\frac{1}{2}\right) + (3)^2\left(\frac{1}{2}\right)$

3. Evaluate $\int x^{\frac{1}{2}}(x^2+x)\,dx$

$$\int x^{\frac{1}{2}}(x^2+x)\,dx = \int\left(x^{5/2}+x^{3/2}\right)dx$$

$$= \frac{2}{7}x^{7/2} + \frac{2}{5}x^{5/2} + C$$

4. Evaluate $\int(x^2+1)\sqrt{x^3+3x}\,dx$

Let $u = x^3+3x$

$du = (3x^2+3)\,dx$

$= 3(x^2+1)\,dx$

$\frac{1}{3}du = (x^2+1)\,dx$

$$\int (x^2+1)\sqrt{x^3+3x}\,dx$$

$$= \int u^{1/2}\left(\tfrac{1}{3}du\right)$$

$$= \frac{1}{3}\int u^{1/2}\,du$$

$$= \frac{1}{3}\left(\frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{2}{9}\left(x^3+3x\right)^{3/2} + C$$

5. Evaluate $\displaystyle\int \cos(5x)\,dx$

Let $u = 5x$

$du = 5\,dx$

$\frac{1}{5}du = dx$

$$\int \cos(5x)\,dx$$

$$= \int \cos(u)\left(\tfrac{1}{5}du\right)$$

$$= \frac{1}{5}\int \cos(u)\,du$$

$$= \frac{1}{5}\sin(u) + C$$

$$= \frac{1}{5}\sin(5x) + C$$

6. Find the average value of the function $f(x) = 2x^2 + 1$ on the interval $[0, 4]$.

$$f_{ave} = \frac{1}{4-0} \int_0^4 (2x^2+1)\,dx$$

$$= \frac{1}{4} \left(\frac{2}{3}x^3 + x \right|_0^4$$

$$= \frac{1}{4} \left(\left(\frac{2}{3}(4)^3 + 4 \right) - 0 \right)$$

$$= \left(\frac{2}{3}\right)(16) + 1$$

$$= \frac{32}{3} + 1 = \frac{35}{3}$$

7. Evaluate $\int \sin(x)\ \cos^3(x)\,dx$.

Let $u = \cos(x)$

$du = -\sin(x)\,dx$

$-du = \sin(x)\,dx$

$$\int \sin(x) \cos^3(x)\,dx$$

$$= \int u^3 (-du)$$

$$= -\int u^3\,du$$

$$= -\frac{1}{4}u^4 + C$$

$$= -\frac{1}{4}\cos^4(x) + C$$

8. Evaluate $\int_{-2}^{2} \frac{x^3}{(x^4+4)^2}\,dx = 0.$

Let $f(x) = \dfrac{x^3}{(x^4+4)^2}$

$$f(-x) = \frac{(-x)^3}{((-x)^4+4)^2} = -\frac{x^3}{(x^4+4)^2}$$

$$= -f(x)$$ Odd

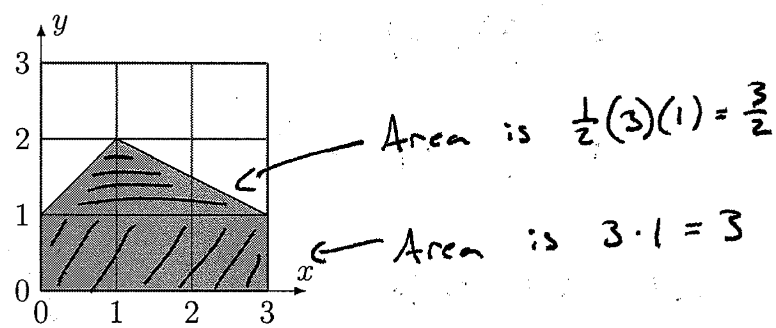
PART II

All problems in Part II are 11 points each.

1. Suppose the graph of a function $y = f(x)$ is triangular, as shown in the plot below.
   (i) Find the value of its integral: $\int_0^3 f(x)\,dx$

   (ii) Let $g(x) = \int_0^x f(t)\,dt$. Is the derivative $g'(\frac{5}{2})$ positive or negative (you must explain your answer!).

   The area of a triangle is $\frac{1}{2}\cdot$ base $\cdot$ height



(i) $\int_0^3 f(x)dx$ is the area under the curve. So

$$\int_0^3 f(x)dx = 3 + \frac{3}{2} = \frac{9}{2}$$

(ii) If $g(x) = \int_0^x f(t)dt$, then Fund. Thm of Calculus says

$g'(x) = f(x)$. Thus, $g'(\frac{5}{2}) = f(\frac{5}{2})$. Since $f(\frac{5}{2}) > 0$, $g'(5/2)$

is positive.

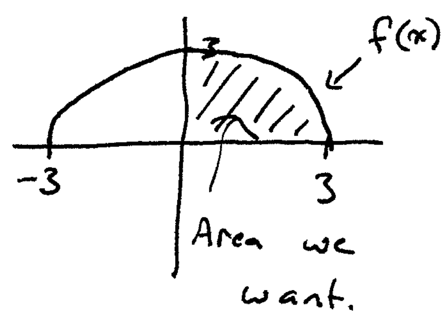
2. Evaluate $\int x\sqrt{x+1}\,dx$

Let $u = x+1$.
$x = u-1$
$du = dx$

$$\int x\sqrt{x+1}\,dx$$
$$= \int (u-1)\sqrt{u}\,du$$
$$= \int \left(u^{3/2} - u^{1/2}\right) du$$
$$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$$
$$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$$

3. Evaluate $\int_0^3 \sqrt{9-x^2}\,dx$. (Hint: consider the graph.)

$f(x) = \sqrt{9-x^2}$ is the equation for the upper half of a circle of radius 3. The area from $x=0$, to $x=3$ covers $1/4$ of the total area of that circle.



Thus, $\int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{4}\pi(3)^2$
$$= \frac{9\pi}{4}$$

4. A particle's acceleration is given by $a(t) = 10t$. At $t = 1$, you measure the velocity of the particle to be 10 and the position of the particle to be 7. Give the function which describes the particle's position in time.

$a(t) = 10t$

$v(t) = 5t^2 + C$

We know $v(1) = 10$ so

$v(1) = 5 + C = 10$ so $C = 5$.

Thus, $v(t) = 5t^2 + 5$

$s(t) = \frac{5}{3}t^3 + 5t + D$

Since $s(1) = 7$, we have

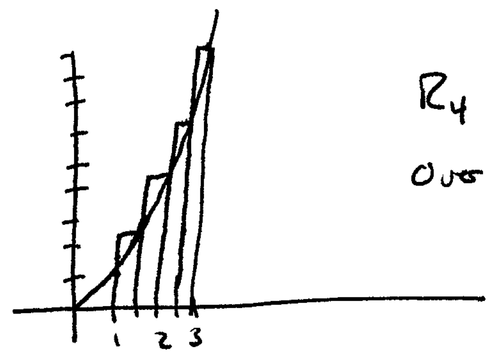
$s(1) = \frac{5}{3} + 5 + D = 7$

$D = \frac{1}{3}$

So, $s(t) = \frac{5}{3}t^3 + 5t + \frac{1}{3}$

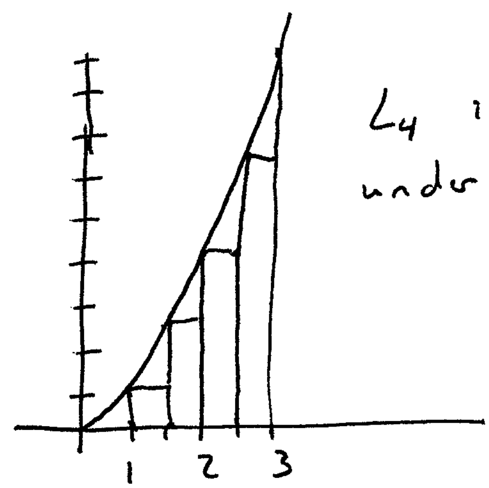
5. **Bonus question worth at most 5 points!** In part I, problem 2 you used a Riemann sum with $n = 4$ terms and the right endpoint rule to approximate $\int_1^3 x^2\,dx$. Is this estimate an upper or lower estimate? As always, you must justify your answer. Find another estimate (also with n=4 terms) so that the true value of the integral is in between the two estimates. What should you use as your best approximate value and what do you know about its error? (You don't need to add and multiply the terms.)

Looking at the graph of x^2 from $x=1$ to $x=3$, we can see R_4 is an overestimate.



R_4 is an overestimate.

Similarly, we can see that the left endpoint rule is an underestimate.



L_4 is an underestimate.

Perhaps the best choice is to use the midpoint rule. The maximum error value would be $R_4 - L_4$.