MA125-6C, CALCULUS I

February 25, 2015

Name (Print last name first): Key

Show all your work and justify your answer!

No partial credit will be given for the answer only!

PART I

You must simplify your answer when possible.
All problems in Part I are 10 points each.

1. Find the derivative of the function $y = f(x) = (\sin(x^3))^2$.

$$f'(x) = 2\left(\sin(x^3)\right)\frac{d}{dx}\left(\sin(x^3)\right)$$

$$= 2\sin(x^3)\cos(x^3)(3x^2)$$

$$= 6x^2\sin(x^3)\cos(x^3)$$

2. Find the derivative of $f(x) = (x^2+1)^2(2x-1)^3$.

$$f'(x) = (x^2+1)^2\frac{d}{dx}\left((2x-1)^3\right) + (2x-1)^3\frac{d}{dx}\left((x^2+1)^2\right)$$

$$= (x^2+1)^2(3)(2x-1)^2(2) + (2x-1)^3(2)(x^2+1)(2x)$$

$$= 6(x^2+1)^2(2x-1)^2 + 4x(2x-1)^3(x^2+1)$$

3. Find the absolute maximum and minimum of the function $y = f(x) = (x^2 - 1)^3$ on the interval $[-2, 2]$.

$f'(x) = 3(x^2-1)^2(2x)$

Critical pts

$x=0,\ x=1,\ x=-1$

Closed interval method since f is continuous

$f(-2) = 27$

$f(2) = 27$

$f(-1) = 0$

$f(1) = 0$

$f(0) = -1$

absolute max is 27

absolute min is -1

4. Verify that the conditions of the Mean Value Theorem hold. Next find the number c which satisfies the conclusion of the Mean Value Theorem for the function $y = f(x) = x^3$ on the interval $[0, 1]$.

x^3 is continuous on $[0,1]$ & differentiable on $(0,1)$.

MVT states there is a c in $(0,1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1-0} = \frac{1-0}{1-0} = 1$$

$f'(x) = 3x^2$

$f'(c) = 1$

$3c^2 = 1$

$c^2 = \frac{1}{3} \Rightarrow c = \frac{1}{\sqrt{3}}$ since c has to be in $(0,1)$.

5. Find all critical numbers of the function $y = f(x) = x + \frac{3}{2}x^{\frac{2}{3}}$.

$$f'(x) = 1 + x^{-1/3} = 1 + \frac{1}{x^{1/3}} = \frac{x^{1/3}+1}{x^{1/3}}$$

$f'(x) = 0$

$x^{1/3} + 1 = 0$

$x^{1/3} = -1$

$x = -1$

$f'(x)$ DNE

$x^{1/3} = 0$

$x = 0$

Critical Pts

$x = -1$

$x = 0$

6. Show that the equation $x^3 + 4x - 7 = 0$ has exactly one real root.

Let $f(x) = x^3 + 4x - 7$. Since f is a polynomial, it is continuous & differentiable everywhere. Further, we can see that

$$f(1) = (1)^3 + 4(1) - 7 = -2 < 0 \quad \&$$
$$f(2) = (2)^3 + 4(2) - 7 = 9 > 0.$$

The IVT then tells us that there is at least one root in the interval $(1, 2)$.

Now, suppose f has two roots, in particular at a & b. Then Rolle's Theorem says there is a c in (a, b) such that $f'(c) = 0$. But $f'(x) = 3x^2 + 4 > 0$ for all x. Thus, f can not have two roots. Thus, f has exactly one real root.

PART II

7. [**10 points**] Federal guidelines mandate that unruly students be kept in pens of volume 100 m^3. These pens are to have a square base of side length l and height h with an open top. (It has been shown that the availability of fresh air calms students.) If the material for the floor costs $\$20/m^2$ and the material for the walls costs $\$5/m^2$, what dimensions will minimize the cost of each pen?

Hint: The volume of such a pen is given by $V = l^2h$. The area of the base is l^2 and the area for **each** wall is lh.

Let C be the cost.

$$C = 20l^2 + 5(4lh)$$
$$= 20l^2 + 20lh$$

$$C(l) = 20l^2 + 20l\left(\frac{100}{l^2}\right)$$
$$= 20l^2 + \frac{2000}{l}, \quad l > 0$$

$$C'(l) = 40l - \frac{2000}{l^2}$$
$$= \frac{40l^3 - 2000}{l^2}$$

C has a critical pt when

$$40l^3 - 2000 = 0.$$
$$40l^3 = 2000$$
$$l^3 = 50$$
$$l = 50^{1/3}$$

Since $C'(l) < 0$ for $0 < l < 50^{1/3}$ & $C'(l) > 0$ for $l > 50^{1/3}$, there is an abs min at $l = 50^{1/3}$.

$$V = l^2h = 100$$
$$\Rightarrow h = \frac{100}{l^2}$$

For $l = 50^{1/3}$

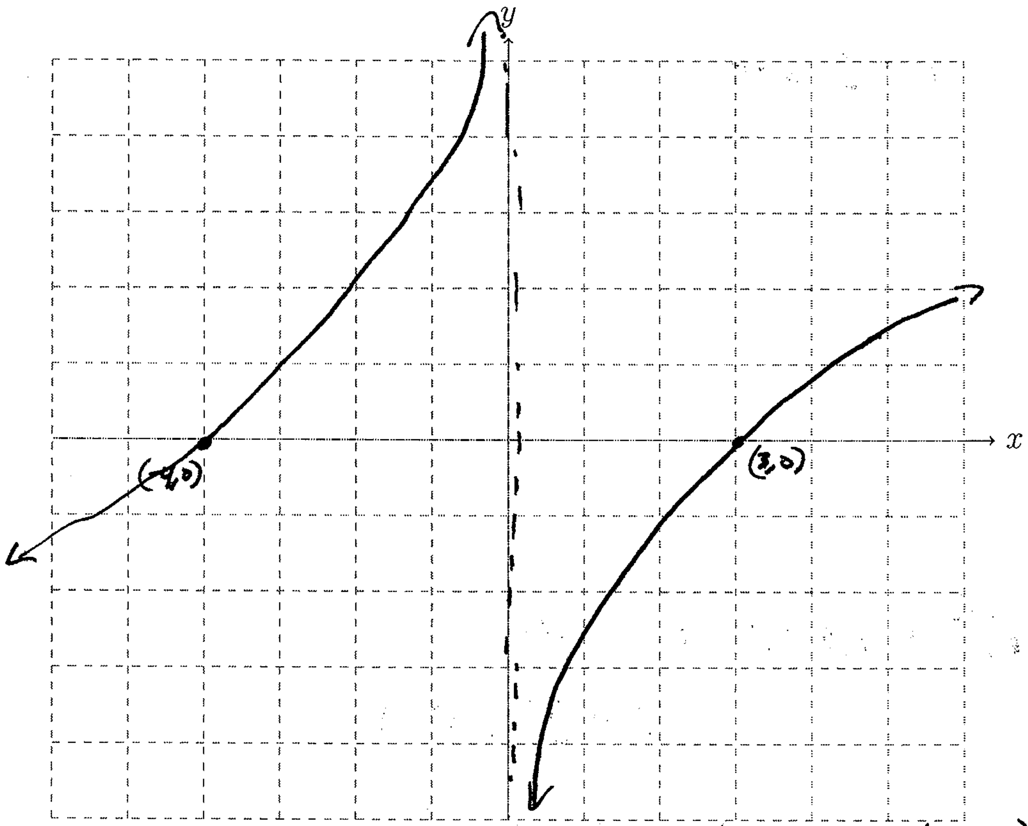
$$h = \frac{100}{(50^{1/3})^2}$$
$$= 2(50)(50^{-2/3})$$
$$= 2(50)^{1/3}$$
$$= 2l$$

Dimensions are $l = 50^{1/3}$ & $h = 2(50)^{1/3}$.

8. [**20 points**] **Use calculus** to graph the function $y = f(x) = \dfrac{x^2 + x - 12}{x}$. Indicate

- x and y intercepts,
- vertical and horizontal asymptotes (if any),
- in/de-creasing; local/absolute max/min (if any).

You must show work to justify your graph and conclusions. You can use decimal numbers to plot points (but mark them with exact values).



x-int

$0 = \dfrac{x^2+x-12}{x}$

$0 = \dfrac{(x+4)(x-3)}{x}$

$x=-4, \ x=3$

y-int

None since $x=0$ is not in the domain.

Vert Asy

$\lim_{x\to 0^-} \dfrac{x^2+x-12}{x} = \infty$

$\lim_{x\to 0^+} \dfrac{x^2+x-12}{x} = -\infty$

Hor Asy

$\lim_{x\to \infty} \dfrac{x^2+x-12}{x} = \infty$

$\lim_{x\to -\infty} \dfrac{x^2+x-12}{x} = -\infty$

$f'(x) = \dfrac{x(2x+1) - (x^2+x-12)(1)}{x^2}$

$= \dfrac{2x^2+x-x^2-x+12}{x^2}$

$= \dfrac{x^2+12}{x^2} > 0$ for $x \neq 0$

So increasing on $(-\infty, 0)\cup(0,\infty)$

9. This question has two parts.

(a) [**6 points**] Find the linearization of $f(x) = x^{\frac{2}{3}}$ at $a = 8$

$L(x) = f(a) + f'(a)(x-a)$

$f(a) = f(8) = (8^{2/3}) = 4$

$f'(x) = \frac{2}{3} x^{-1/3}$

$f'(a) = f'(8) = \frac{2}{3}(8)^{-1/3} = \frac{2}{3}\frac{1}{2} = \frac{1}{3}$

$L(x) = 4 + \frac{1}{3}(x-8)$

(b) [**4 points**] Use this linearization to find the approximate value of $(8.05)^{\frac{2}{3}}$.

$(8.05)^{2/3} = f(8.05) \approx L(8.05)$

$= 4 + \frac{1}{3}(8.05 - 8)$

$= 4 + \frac{1}{3}(0.05)$

$= 4 + 0.02$

$= 4.02$