MA 125-6C, CALCULUS I

Test 1, February 4, 2015

Name (Print last name first): Key

Show all your work and justify your answer!

No partial credit will be given for the answer only!

PART I

You must simplify your answer when possible.
All problems in Part I are 7 points each.

1. Use the **definition** of the derivative to show that if $f(x) = x^2$, then $f'(x) = 2x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h)}{h}$$

$$= \lim_{h\to 0} 2x + h$$

$$= 2x$$

2. Evaluate $\lim_{x\to 1} \dfrac{x^2+3x-1}{x+2}$

By the direct substitution property,

$$\lim_{x\to 1} \frac{x^2+3x-1}{x+2} = \frac{(1)^2+3(1)-1}{1+2} = \frac{3}{3} = \boxed{1}$$

3. Evaluate $\lim_{x\to\infty} \frac{3x^3+2x^2+x-1}{2x^4+4x^3-x^2-6}$

$$\lim_{x\to\infty} \frac{3x^3+2x^2+x-1}{2x^4+4x^3-x^2-6} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{3+\frac{2}{x}^{\to 0}+\frac{1}{x^2}^{\to 0}-\frac{1}{x^3}^{\to 0}}{2x+4-\frac{1}{x}^{\to 0}-\frac{6}{x^3}^{\to 0}}$$

$$= \boxed{0}$$

4. Given that $y = f(x) = (x^2+2)(x^3-1)$, find $f'(x)$.

$f(x) = x^5 - x^2 + 2x^3 - 2$

$f'(x) = 5x^4 - 2x + 6x^2$

$\boxed{= 5x^4 + 6x^2 - 2x}$

5. Given that $y = f(x) = x^3 \sec(x)$, find $f'(x)$.

$f'(x) = x^3 \frac{d}{dx}(\sec(x)) + \sec(x) \frac{d}{dx}(x^3)$

$\boxed{= x^3 \tan(x)\sec(x) + 3x^2 \sec(x)}$

6. Given that $y = f(x) = \frac{1+\cos(x)}{x^2-1}$, find $f'(x)$.

$$f'(x) = \frac{(x^2-1)\frac{d}{dx}(1+\cos(x)) - (1+\cos(x))\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)(-\sin(x)) - (1+\cos(x))(2x)}{(x^2-1)^2}$$

$$= \boxed{\frac{\sin(x) - x^2\sin(x) - 2x - 2x\cos(x)}{(x^2-1)^2}}$$

7. Find the equation of the tangent line to the graph of $y = f(x) = \sin(x)$ at the point $x = \pi$.

$f'(x) = \cos(x)$

$m = f'(\pi) = \cos(\pi) = -1$

$(x_0, y_0) = (\pi, f(\pi)) = (\pi, 0)$

$y - y_0 = m(x - x_0)$

$y - 0 = -1(x - \pi)$

$\boxed{y = -x + \pi}$

8. Evaluate $\lim_{x\to 3} \frac{x^2-9}{x^2-x-6}$

$$\lim_{x\to 3} \frac{x^2-9}{x^2-x-6} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)(x+2)} = \lim_{x\to 3} \frac{x+3}{x+2} = \boxed{\frac{6}{5}}$$

9. If the **position** of a particle, at time t, is given by $s(t) = t^3 - 4t$, find the **acceleration** $a(t)$ at time $t = 1$. Is the **velocity** $v(t)$ increasing or decreasing at this time? Justify your answer.

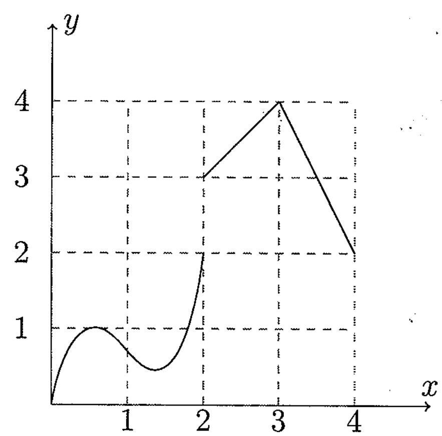
$v(t) = s'(t) = 3t^2 - 4$

$a(t) = v'(t) = 6t$

Acceleration at $t=1$ is $a(1) = 6(1) = 6$.

Since $a(1)$ is positive, velocity is increasing.

10. Given the graph of the function below, state (a) where it is continuous and (b) where the derivative exist.



Continuous on $(0,2) \cup (2,4)$

Differentiable on $(0,2) \cup (2,3) \cup (3,4)$

PART II

All problems in Part II are 10 points each.

1. Evaluate

(a) $\lim_{x\to 0} \frac{\sin(3x)}{2x}$

$$= \lim_{x\to 0} \frac{3}{3}\cdot\frac{\sin(3x)}{2x}$$

$$= \lim_{x\to 0} \frac{3}{2}\cdot\frac{\sin(3x)}{3x}$$

$$= \frac{3}{2}\lim_{x\to 0} \frac{\sin(3x)}{3x}$$

$$= \frac{3}{2}(1) = \boxed{\frac{3}{2}}$$

(b) $\lim_{x\to 2} \frac{3x}{(2x-4)^2}$

$$\lim_{x\to 2} \frac{3x}{(2x-4)^2} = \lim_{x\to 2}\left((3x)\frac{1}{(2x-4)^2}\right) = \left(\lim_{x\to 2} 3x\right)\left(\lim_{x\to 2}\frac{1}{(2x-4)^2}\right)$$

$$= \boxed{\infty}$$

$\lim_{x\to 2} 3x = 6$

$\lim_{x\to 2} \frac{1}{(2x-4)^2} = \infty$

Since

$\lim_{x\to 2^+} \frac{1}{(2x-4)^2} = \infty$

& $\lim_{x\to 2^-} \frac{1}{(2x-4)^2} = \infty$

2. Evaluate $\lim_{x\to\infty} \frac{\cos(x)}{x}$ *Hint: The Squeeze Theorem can be very useful.*

Since $-1 \le \cos(x) \le 1$, we have

$$-\frac{1}{x} \le \frac{\cos(x)}{x} \le \frac{1}{x}.$$

Since $\lim_{x\to\infty} -\frac{1}{x} = 0$ & $\lim_{x\to\infty} \frac{1}{x} = 0$, the squeeze theorem gives

$$\lim_{x\to\infty} \frac{\cos(x)}{x} = 0.$$

3. Below you are given the graph of the **derivative** $f'(x)$ of a function $y = f(x)$.

(a) State the x-coordinates of all points where the graph of $y = f(x)$ has a horizontal tangent line.

Since horizontal tangent lines mean the derivative is zero, we have a horizontal tangent line at

$x = -4.6, -1.3, 3.5$

(b) State all x-values of points where the rate of change of the function $y = f(x)$ is maximal.

This corresponds to the largest value, in absolute value, that the derivative takes. Thus, $x = 1$.

Graph of the **derivative** $f'(x)$