MA125-8B, CALCULUS I

September 25, 2014

Name (Print last name first): Key

TEST I

Show all your work and justify your answer!

No partial credit will be given for the answer only!

PART I

You must simplify your answer when possible.
All problems in Part I are 6 points each.

1. Show, **using the definition**, that the derivative of $f(x) = 2x^2 + x$ is $4x + 1$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h)^2 + (x+h) - (2x^2 + x)}{h}$$

$$= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 + x + h - 2x^2 - x}{h}$$

$$= \lim_{h \to 0} \frac{4xh + 2h^2 + h}{h}$$

$$= \lim_{h \to 0} 4x + 2h + 1$$

$$= 4x + 1$$

2. Evaluate $\lim_{x \to -2} \frac{x^2 - x - 6}{x^2 - 4}$

$$\lim_{x \to -2} \frac{x^2 - x - 6}{x^2 - 4}$$

$$= \lim_{x \to -2} \frac{(x-3)(x+2)}{(x-2)(x+2)}$$

$$= \lim_{x \to -2} \frac{x-3}{x-2}$$

$$= \frac{-2-3}{-2-2}$$

$$= \frac{-5}{-4} = \boxed{\frac{5}{4}}$$

3. Evaluate $\lim_{x \to \frac{\pi}{2}} \sqrt{\sin(x) + 15}$

$$\lim_{x \to \pi/2} \sqrt{\sin(x) + 15}$$

$$= \sqrt{\lim_{x \to \pi/2} (\sin(x) + 15)}$$

$$= \sqrt{\sin(\pi/2) + 15}$$

$$= \sqrt{1 + 15} = \sqrt{16} = \boxed{4}$$

4. Find the derivative of $y = f(x) = x^2 \tan(x)$

$$f'(x) = x^2 \frac{d}{dx}(\tan(x)) + \tan(x) \frac{d}{dx}(x^2)$$

$$= \boxed{x^2 \sec^2(x) + 2x \tan(x)}$$

5. Find the derivative of $y = f(x) = \frac{x^3 + x}{x^{\frac{3}{2}}}$

$$f(x) = x^{3/2} + x^{-1/2}$$

$$\boxed{f'(x) = \frac{3}{2} x^{1/2} - \frac{1}{2} x^{-3/2}}$$

6. Find the derivative of $y = f(x) = \dfrac{\sqrt{x}-1}{\sqrt{x}+1}$

$$f'(x) = \frac{(\sqrt{x}+1)\frac{d}{dx}(\sqrt{x}-1) - (\sqrt{x}-1)\frac{d}{dx}(\sqrt{x}+1)}{(\sqrt{x}+1)^2}$$

$$= \frac{(\sqrt{x}+1)\left(\frac{1}{2}x^{-1/2}\right) - (\sqrt{x}-1)\left(\frac{1}{2}x^{-1/2}\right)}{(\sqrt{x}+1)^2}$$

$$= \frac{\left(\frac{1}{2}x^{-1/2}\right)(\sqrt{x}+1-\sqrt{x}+1)}{(\sqrt{x}+1)^2}$$

$$= \frac{\left(\frac{1}{2}x^{-1/2}\right)(2)}{(\sqrt{x}+1)^2} = \boxed{\frac{x^{-1/2}}{(\sqrt{x}+1)^2}}$$

7. Find the derivative of $y = f(x) = \sqrt[3]{\sin(x) - \cos(x)}$

$$f(x) = (\sin(x) - \cos(x))^{1/3}$$

$$f'(x) = \frac{1}{3}(\sin(x) - \cos(x))^{-2/3} \frac{d}{dx}(\sin(x) - \cos(x))$$

$$= \frac{1}{3}(\sin(x) - \cos(x))^{-2/3}(\cos(x) + \sin(x))$$

8. Find the equation of the tangent line to the graph of $y = f(x) = 2x^3 + x$ at the point $x = 2$.

$$f'(x) = 6x^2+1$$

$$f'(2) = 6(2)^2+1$$

$$= 25$$

$$f(2) = 2(2)^3 + (2)$$

$$= 16+2$$

$$= 18$$

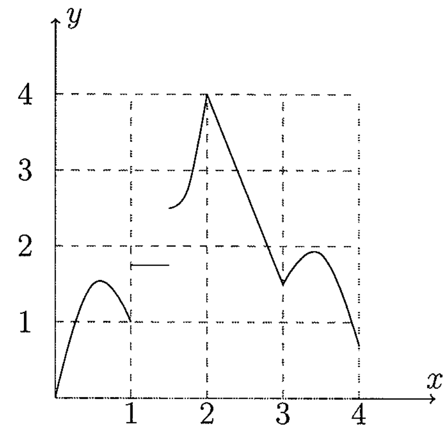
Point on the graph $(2, f(2)) = (2, 18)$

$$y - 18 = 25(x-2)$$

$$y = 25x - 50 + 18$$

$$\boxed{y = 25x - 32}$$

9. Given the graph of the function below, state (a) where it is continuous and (b) where the derivative exists.



a) Continuous on

$(0,1) \cup (1,1.5) \cup (1.5,4)$

b) Differentiable on

$(0,1) \cup (1,1.5) \cup (1.5,2) \cup (2,3) \cup (3,4)$

10. Evaluate $\lim_{x\to\infty} \dfrac{2x^2-3x+4}{x^3+5x^2+x-100}$

$$\lim_{x\to\infty} \frac{2x^2-3x+4}{x^3+5x^2+x-100} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\frac{2}{x} - \frac{3}{x^2} + \frac{4}{x^3}}{1 + \frac{5}{x} + \frac{1}{x^2} - \frac{100}{x^3}}$$

$$= \frac{0}{1} = \boxed{0}$$

PART II

Points for each problem are as indicated.

1. **[10 points]** Find the x-coordinates of all points where the graph of $y = f(x) = (2-x)^4(4x+1)^3$ has a horizontal tangent line

Find where $f'(x) = 0$.

$$f'(x) = (2-x)^4 \frac{d}{dx}\left((4x+1)^3\right) + (4x+1)^3 \frac{d}{dx}\left((2-x)^4\right)$$

$$= (2-x)^4(3)(4x+1)^2(4) + (4x+1)^3(4)(2-x)^3(-1)$$

$$= 4(2-x)^3(4x+1)^2\left(3(2-x) - (4x+1)\right)$$

$$= 4(2-x)^3(4x+1)^2(6-3x-4x-1)$$

$$= 4(2-x)^3(4x+1)^2(5-7x)$$

$$\boxed{x = 2, -\frac{1}{4}, \frac{5}{7}}$$

2. [**10 points**] If the position of a particle is given by $S(t) = 5t^2 + \sin(t)$, find the velocity and acceleration at time $t = \pi/2$.

$v(t) = S'(t) = 5(2t) + \cos(t) = 10t + \cos(t)$

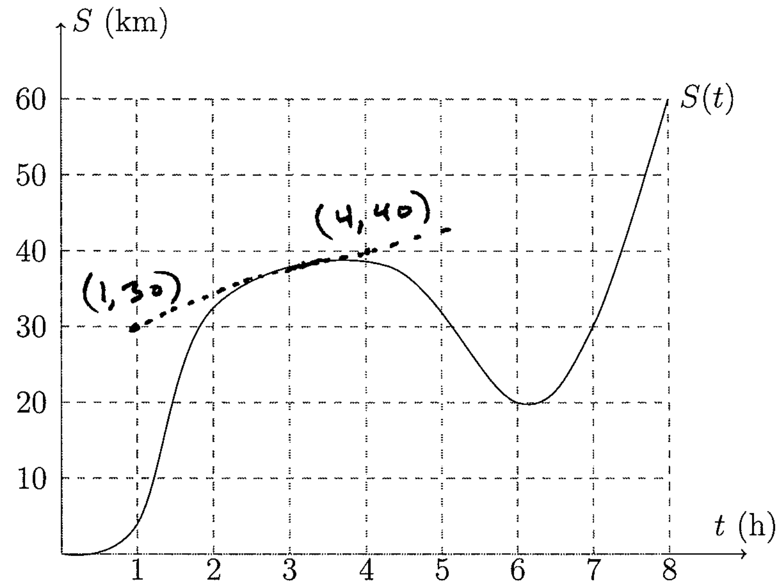
$v(\pi/2) = 10(\pi/2) + \cos(\pi/2)$

$= \boxed{5\pi}$

$a(t) = v'(t) = S''(t) = 10 - \sin(t)$

$a(\pi/2) = 10 - \sin(\pi/2) = 10 - 1 = \boxed{9}$

3. [**10 points**] A rail road car travels along a straight track. The graph below gives its position (in km) as a function of time (in h).



(a) Estimate the velocity at time $t = 3$. [Show your work!!]

slope of tangent line $m = \frac{40-30}{4-1} = \boxed{\frac{10}{3}}$

(b) At the above time was the velocity increasing or decreasing? [Explain!!]

Around $t=3$, the slope of the tangent line is decreasing. Thus, the velocity is decreasing.

(c) At the above time was the car accelerating or decelerating? [Explain!]

Velocity decreasing means the car was decelerating.

4. You can (and should) use your calculator in the following problem (but do not use the derivative of this function, even if you know it; in the latter case you can of course check to see if your answer is reasonable). Suppose that $C(x) = 3\ln(1+x)$ is the cost of producing x-items. We are interested in the derivative of this function.

(a) [**3 points**] Give a verbal description of the meaning (in terms of cost) of the derivative $C'(1000)$.

$C'(1000)$ is the rate of change of the cost of producing more units when you are producing 1000 units.

(b) [**4 points**] Using the definition of the derivative, write $C'(2)$ as a limit ($\lim_{h\to 0}$ ......) . (Use the definition of $C(x)$ given above.)

$$C'(2) = \lim_{h\to 0} \frac{C(2+h) - C(2)}{h}$$

$$= \lim_{h\to 0} \frac{3\ln(3+h) - 3\ln(3)}{h}$$

(c) [**3 points**] Estimate $C'(2)$ using your calculator (you only need to use $h = \pm\frac{1}{10}$ to get your approximate answer) .

$h = \frac{1}{10}$

$$\frac{3\ln(3.1) - 3\ln(3)}{0.1}$$

$= 0.983695$

$h = -\frac{1}{10}$

$$\frac{3\ln(2.9) - 3\ln(3)}{-0.1}$$

$= 1.01705$

So, $C'(2) \approx 1.$